EQUIVARIANCE AND IMPRIMITIVITY FOR DISCRETE HOPF C^*-COACTIONS

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ABSTRACT. Let U, V, and W be multiplicative unitaries coming from discrete Kac systems such that W is an amenable normal submultiplicative unitary of V with quotient U. We define notions for right-Hilbert bimodules of coactions of S_V and $\hat{S}_V$, their restrictions to S_W and $\hat{S}_U$, their dual coactions, and their full and reduced crossed products. If $N(A)$ denotes the imprimitivity bimodule associated to a coaction δ of S_V on a C^*-algebra A by Ng's imprimitivity theorem, we prove that for a suitably nondegenerate injective right-Hilbert bimodule coaction of S_V on ${}_AX_B$, the balanced tensor products $N(A)\otimes_{A\times\hat{S}_W}({}_AX_B\times\hat{S}_W)$ and $({}_AX_B\times\hat{S}_V\times_r S_U)\otimes_{B\times\hat{S}_V\times_r S_U}N(B)$ are isomorphic right-Hilbert $A\times\hat{S}_V\times_r S_U$ – $B\times\hat{S}_W$ bimodules. This can be interpreted as a natural equivalence between certain crossed-product functors.

1. Introduction

Since Baaj and Skandalis introduced multiplicative unitaries in [2] as a generalization of locally compact groups, and proved a duality theorem ([2, Théorème 7.5]) for crossed products by coactions of the associated Hopf C^*-algebras, there has been much interest in extending other results for group actions and coactions to this context. Recently Ng ([18]) has defined notions of sub- and quotient multiplicative unitaries, and has proved that for multiplicative unitaries U, V, and W coming from discrete Kac systems such that W is an amenable normal submultiplicative unitary of V with quotient U, and for any injective nondegenerate coaction δ of S_V on a C^*-algebra A, the iterated crossed product $A\times_\delta\hat{S}_V\times_{\hat{\delta}|,r}S_U$ is Morita equivalent to $A\times_{\delta|}\hat{S}_W$ ([18, Theorem 3.4]). This is an analog both of Green's celebrated imprimitivity theorem ([9]), which implies that for an action α of a group G on A and a closed normal subgroup N of G, $A\times_\alpha G\times_{\hat{\alpha}|}G/N$ is Morita equivalent to $A\times_{\alpha|}N$, and of Mansfield's imprimitivity theorem ([15]) for coactions (as generalized to non-amenable groups in [11]), which provides a Morita equivalence between $A\times_\delta G\times_{\hat{\delta}|,r}N$ and $A\times_{\delta|}G/N$ for a coaction δ (satisfying a mild condition) of G on A and any closed normal subgroup N of G. For discrete multiplicative unitaries, Ng's theorem generalizes Baaj-Skandalis

Date: August 10, 1998.

1991 *Mathematics Subject Classification.* 46L55, secondary 22D25.

This research was partially supported by the Australian Research Council and by the National Science Foundation (under Grant No. DMS9401253).

duality (ignoring differences between full and reduced crossed products) in the same way that Green's theorem generalizes Imai-Takai-Takesaki duality ([10]), and Mansfield's theorem generalizes the duality of Katayama ([13]).

Now the significance of Green's theorem is that his imprimitivity bimodule may be viewed as a Hilbert $A \times_{\alpha|} N$-module with a nondegenerate left action of $A \times_\alpha G$ by adjointable operators, and thus allows induction of representations from $A \times_{\alpha|} N$ to $A \times_\alpha G$ via Rieffel's framework ([21]). Similarly, Mansfield's bimodule allows induction of representations from $A \times_{\delta|} G/N$ to $A \times_\delta G$. The representation-inducing processes arising from these bimodules, and their interactions with one another, have received much attention lately (see [3, 8, 5, 12, 20]), and the method that has evolved is to work with the bimodules that implement the inducing maps on representations, rather than with those inducing maps themselves. We call the bimodules involved *right-Hilbert bimodules*; they are essentially imprimitivity bimodules ${}_KX_B$ together with nondegenerate homomorphisms of A into $M(K)$.

The equivariant right-Hilbert bimodules — that is, those right-Hilbert bimodules ${}_AX_B$ which carry compatible actions or coactions of a group G — turn out to be closely related to imprimitivity theorems. In work with S. Echterhoff and I. Raeburn which is currently in preparation we have shown, for example, that Green's imprimitivity theorem can be viewed as a natural equivalence between the crossed product functors $(A, G, \alpha) \mapsto A \times_\alpha G \times_{\hat\alpha|} G/N$ and $(A, G, \alpha) \mapsto A \times_{\alpha|} N$ defined on a category whose objects are C^*-algebras with actions of G and whose morphisms $(A, \alpha) \to (B, \beta)$ are (isomorphism classes) of equivariant right-Hilbert $A - B$ bimodules ([4]).

In this paper, we show that Ng's imprimitivity theorem is similarly compatible with equivariant right-Hilbert bimodules. To do so, we must first develop a theory of coactions of Hopf C^*-algebras S_V and $\hat S_V$ on right-Hilbert bimodules, and their crossed products; this is done as efficiently as possible in Section 2 by building for the most part on Ng's imprimitivity bimodule apparatus ([16]). In Section 3, we review Ng's fixed-point theorem ([18, Proposition 2.11]), since it provides the construction of the bimodule which appears in his imprimitivity theorem. Here we prove two lemmas relating Ng's bimodule to the linking algebra and standard right-Hilbert bimodule (see below) constructions we use in proving our main theorem.

In the final section, we prove our main result: for U, V, and W as in Ng's theorem, and for a suitably nondegenerate injective right-Hilbert bimodule coaction of S_V on ${}_AX_B$,

$$N(A) \otimes_{A \times \hat S_W} X \times \hat S_W \cong X \times \hat S_V \times_r S_U \otimes_{B \times \hat S_V \times_r S_U} N(B)$$

as right-Hilbert $A \times \hat S_V \times_r S_U - B \times \hat S_W$ bimodules, where $N(A)$ denotes Ng's $A \times \hat S_V \times_r S_U - A \times \hat S_W$ imprimitivity bimodule (and similarly for $N(B)$). As discussed above for group actions, this should give a natural equivalence between certain crossed-product functors, although we don't formalize this in the present paper. (Part of our point here is that any reasonable imprimitivity theorem should be compatible with equivariant right-Hilbert

bimodules, and that the proof of this, following the same strategy we use in the proof of Theorem 4.1, should be relatively straightforward.) Our theorem should have implications for induced and restricted representations of crossed products by Hopf C^*-algebras, and for equivariant KK-theory as in [1].

A substantial amount of this research was conducted while the first author visited the second at Arizona State University, and on another occasion while both authors visited the University of Newcastle. We particularly thank Iain Raeburn in Newcastle for his hospitality.

Preliminaries

For compatibility with Ng's work on imprimitivity bimodules, we define a *right-Hilbert A – B bimodule* over C^*-algebras A and B to be an imprimitivity bimodule ${}_KX_B$ together with a nondegenerate homomorphism of A into $M(K)$. If X is a full Hilbert B-module (cf. [14]), then X is a $\mathcal{K}_B(X)$ – B imprimitivity bimodule, and $M(\mathcal{K}_B(X)) = \mathcal{L}_B(X)$, so this is the same as having a nondegenerate action of A by adjointable operators on X. Note that K itself becomes a right-Hilbert $A - K$ bimodule by using the natural $K - K$ imprimitivity bimodule structure on K; we call this a *standard* right-Hilbert bimodule. We have the decomposition ${}_AX_B \cong {}_AK \otimes_K X_B$ of any right-Hilbert bimodule as a balanced tensor product of a standard bimodule and an imprimitivity bimodule. We use the conventions of [6] regarding multiplier bimodules, linking algebras, and homomorphisms of imprimitivity bimodules.

Let (S, δ_S) be a Hopf C^*-algebra, and let $\delta\colon A \to M(A \otimes S)$ be a coaction of S on a C^*-algebra A, as in [2, Definition 0.2]. The coaction δ is called *nondegenerate* if $\overline{\operatorname{span}}\{\delta_A(A)(1 \otimes S)\} = A \otimes S$. A *covariant pair* for (A, S, δ) on a C^*-algebra B consists of a nondegenerate homomorphism $\theta\colon A \to M(B)$ and a unitary corepresentation $u \in M(B \otimes S)$ of S such that

$$(\theta \otimes \mathrm{id}) \circ \delta(a) = \mathrm{Ad}(u)(\theta(a) \otimes 1)$$

for each $a \in A$ ([17, Definition 2.8]). The *full crossed product* for (A, S, δ) is a C^*-algebra $A \times_\delta \hat{S}$ together with a universal covariant pair (j, v) for (A, S, δ) on $A \times_\delta \hat{S}$ ([17, Definition 2.11(b)]). If $S = \hat{S}_V$ for a multiplicative unitary V coming from a Kac system (see below), we write $A \times_\delta S_V$ (with no hat) for $A \times_\delta \hat{S}$.

Let $V \in \mathcal{L}(H \otimes H)$ be a regular multiplicative unitary as in [2]. We let L and ρ denote the maps of $\mathcal{L}(H)_*$ into $\mathcal{L}(H)$ defined by

$$L(\omega) = (\omega \otimes \mathrm{id})(V) \quad \text{and} \quad \rho(\omega) = (\mathrm{id} \otimes \omega)(V);$$

then we have the associated reduced Hopf C^*-algebras

$$S_V = \overline{\operatorname{span}}\{L(\omega) \mid \omega \in \mathcal{L}(H)_*\} \quad \text{and} \quad \hat{S}_V = \overline{\operatorname{span}}\{\rho(\omega) \mid \omega \in \mathcal{L}(H)_*\}$$

with comultiplications δ_V and $\hat{\delta}_V$, respectively, given by

$$\delta_V(x) = V(x \otimes 1)V^* \quad \text{and} \quad \hat{\delta}_V(y) = V^*(1 \otimes y)V$$

([2, Théorème 3.8]). The corresponding full Hopf C^*-algebras are denoted $(S_V)_p$ and $(\hat{S}_V)_p$, but their comultiplications are still denoted δ_V and $\hat{\delta}_V$ ([2, Corollaire A.6]). We view L both as a faithful representation of S_V and as a nondegenerate representation of $(S_V)_p$ on $\mathcal{L}(H)$; similarly, we view ρ as a map on both $\hat{S}_V$ and $(\hat{S}_V)_p$ (cf. [17, Proposition 1.16(i)]).

The unitary corepresentations $u \in M(B \otimes S_V)$ of S_V are in bijective correspondence with the nondegenerate homomorphisms $\nu\colon (\hat{S}_V)_p \to M(B)$ ([17, Lemma2.6]). If (A, S_V, δ) is a coaction, by [17, Remark 2.12(b)] we have

$$A \times_\delta \hat{S}_V = \overline{\operatorname{span}}\{j(a)\mu(y) \mid a \in A, y \in (\hat{S}_V)_p\},$$

where $\mu\colon (\hat{S}_V)_p \to M(A \times_\delta \hat{S}_V)$ is the nondegenerate homomorphism corresponding to v. For every pair of homomorphisms $\theta\colon A \to M(B)$ and $\nu\colon (\hat{S}_V)_p \to M(B)$ coming from a covariant pair (θ, u), there is (by definition of the crossed product) a unique nondegenerate homomorphism $\theta \times \nu\colon A \times_\delta \hat{S}_V \to M(B)$ such that

$$(\theta \times \nu) \circ j = \theta \quad \text{and} \quad (\theta \times \nu) \circ \mu = \nu,$$

and the latter condition is equivalent to

$$((\theta \times \nu) \otimes \mathrm{id})(v) = u.$$

Let $\pi_L = (\mathrm{id} \otimes L) \circ \delta\colon A \to \mathcal{L}_A(A \otimes H)$. Then the *reduced crossed product* ([17, Definition 2.11(a)]) is

$$A \times_{\delta,r} \hat{S}_V = C^*\big(\{\pi_L(a)(1 \otimes \rho(\omega)) \mid a \in A, \omega \in \mathcal{L}(H)_*\}\big) \subseteq \mathcal{L}_A(A \otimes H).$$

The reduced crossed product by a coaction δ_p of $(S_V)_p$ is defined similarly, and we have

$$A \times_{\delta_p,r} (\hat{S}_V)_p \cong A \times_{\delta,r} \hat{S}_V,$$

where the coaction $\delta = (\mathrm{id} \otimes L) \circ \delta_p$ of S_V is the *reduction* of δ_p ([17, Proposition 2.14]). There is a *dual coaction* $\hat{\delta}$ of $(\hat{S}_V)_p$ on the full crossed product $A \times_\delta \hat{S}_V$ which satisfies

$$\hat{\delta}(j(a)\mu(y)) = (j(a) \otimes 1)(\mu \otimes \mathrm{id})(\hat{\delta}_V(y))$$

for all $a \in A$, $y \in (\hat{S}_V)_p$; we also denote its reduction by $\hat{\delta}$ ([19, Proposition 2.13]).

Now suppose V comes from a Kac system (H, V, U) ([2, Définition 6.4]). For any coaction δ of S_V on A, we have

(1.1)
$$A \times_{\delta,r} \hat{S}_V = \overline{\operatorname{span}}\{\pi_L(a)(1 \otimes \rho(\omega)) \mid a \in A, \omega \in \mathcal{L}(H)_*\} \subseteq \mathcal{L}_A(A \otimes H).$$

Similarly, for any coaction δ of $\hat{S}_V$ on A, we denote the reduced crossed product by $A \times_{\delta,r} S_V$ ([2, Définition 7.1]), and we have

(1.2)
$$A \times_{\delta,r} S_V = \overline{\operatorname{span}}\{\hat{\pi}_\lambda(a)(1 \otimes L(\omega)) \mid a \in A, \omega \in \mathcal{L}(H)_*\} \subseteq \mathcal{L}_A(A \otimes H),$$

where $\lambda = \mathrm{Ad}(U) \circ \rho$ and $\hat{\pi}_\lambda = (\mathrm{id} \otimes \lambda) \circ \delta$ ([2, Lemme 7.2]).

It is important to note that for any Kac system (H,V,U), $(H,\hat{V},U)$ is also a Kac system ([2, Proposition 6.5]), where $\hat{V}=\Sigma(U\otimes 1)V(U\otimes 1)\Sigma$ and Σ denotes the flip operator on $H\otimes H$, and that then $(\hat{S}_V,\hat{\delta}_V)\cong(S_{\hat{V}},\delta_{\hat{V}})$ as a consequence of [2, Proposition 6.7]. Hence, for any coaction $(A,\hat{S}_V,\delta)$ there is a coaction $(A,S_{\hat{V}},\delta')$ such that $A\times_\delta S_V\cong A\times_{\delta'}\hat{S}_{\hat{V}}$ and $A\times_{\delta,r}S_V\cong A\times_{\delta',r}\hat{S}_{\hat{V}}$. Thus any results about crossed products by coactions of S_V always yield analogous results for coactions of $\hat{S}_V$: for example, Equation (1.2) above can be derived from Equation (1.1) by replacing V by $\hat{V}$.

Let V be a regular multiplicative unitary, and let $\psi\colon A\to M(B)$ be a nondegenerate homomorphism which is *equivariant* for coactions δ_A and δ_B of S_V; i.e. such that

$$\delta_B\circ\psi=(\psi\otimes\mathrm{id})\circ\delta_A.$$

If (j_B,v_B) is the universal covariant pair for (B,S_V,δ_B) on $B\times\hat{S}_V$, then $(j_B\circ\psi,v_B)$ is a covariant pair for (A,S_V,δ_A), so we get a nondegenerate homomorphism $\psi\times\hat{S}_V=(j_B\circ\psi)\times\mu_B\colon A\times\hat{S}_V\to M(B\times\hat{S}_V)$, where $\mu_B\colon(\hat{S}_V)_p\to M(A\times\hat{S}_V)$ corresponds to v_B as in [17, Lemma 2.6]. If ψ is equivariant for coactions of $\hat{S}_V$ on A and B, we likewise get a nondegenerate homomorphism $\psi\times S_V\colon A\times S_V\to M(B\times S_V)$.

The analogous result for reduced crossed products, which we will need in order to define the reduced right-Hilbert bimodule crossed products in the next section, requires a bit more work:

Lemma 1.1. *Let V be a regular multiplicative unitary on a Hilbert space H, and let δ_A and δ_B be coactions of S_V on C^*-algebras A and B. Suppose also that $\psi\colon A\to M(B)$ is a $\delta_A-\delta_B$ equivariant nondegenerate homomorphism. Then there exists a nondegenerate homomorphism $\psi\times_r\hat{S}_V\colon A\times_{\delta_A,r}\hat{S}_V\to M(B\times_{\delta_B,r}\hat{S}_V)$ such that*

$$(\psi\times_r\hat{S}_V)\big(\pi_L^A(a)(1\otimes\rho(y))\big)=\pi_L^B(\psi(a))(1\otimes\rho(y))\tag{1.3}$$

for $a\in A$ and $y\in\hat{S}_V$.

Proof. As in the proof of [2, Théorème 7.5], $A\times_r\hat{S}_V$ acts nondegenerately on $A\otimes H$, and therefore on $(A\otimes H)\otimes_A B$, where B is the standard right-Hilbert $A-B$ bimodule arising from ψ. It is straightforward to check that the map $\Phi\colon(A\otimes H)\otimes_A B\to B\otimes H$ determined by

$$\Phi((a\otimes\xi)\otimes_A b)=\psi(a)b\otimes\xi$$

is a Hilbert B-module isomorphism; thus $\mathcal{L}_B((A\otimes H)\otimes_A B)\cong\mathcal{L}_B(B\otimes H)$, so we obtain a nondegenerate homomorphism $\psi\times_r\hat{S}_V\colon A\times_r\hat{S}_V\to\mathcal{L}_B(B\otimes H)$ characterized by

$$(\psi\times_r\hat{S}_V)\big(\pi_L^A(a)(1\otimes\rho(y))\big)\big(\Phi((c\otimes\xi)\otimes_A b)\big)=\Phi\big(\pi_L^A(a)(1\otimes\rho(y))((c\otimes\xi)\otimes_A b)\big)$$

for $a,c\in A$, $y\in\hat{S}_V$, $\xi\in H$, and $b\in B$.

Now for a, c, ξ, and b as above, factor $\xi = L(x)\eta$ for some $x \in S_V$ and $\eta \in H$, and choose $a_i \in A$ and $x_i \in S_V$ such that $\delta_A(a)(1 \otimes x) \approx \sum_i^n a_i \otimes x_i$. Then we have

$$
\begin{aligned}
&(\psi \times_r \hat{S}_V)(\pi_L^A(a))\big(\Phi((c \otimes \xi) \otimes_A b)\big) \\
&\quad = \Phi\big(\pi_L^A(a)(c \otimes \xi) \otimes_A b\big) \\
&\quad = \Phi\big((\mathrm{id} \otimes L) \circ \delta_A(a)(c \otimes L(x)\eta) \otimes_A b\big) \\
&\quad = \Phi\big((\mathrm{id} \otimes L)(\delta_A(a)(1 \otimes x))(c \otimes \eta) \otimes_A b\big) \\
&\quad \approx \sum_i^n \Phi\big((\mathrm{id} \otimes L)(a_i \otimes x_i)(c \otimes \eta) \otimes_A b\big) \\
&\quad = \sum_i^n \Phi\big((a_i \otimes L(x_i))(c \otimes \eta) \otimes_A b\big) \\
&\quad = \sum_i^n \Phi\big((a_i c \otimes L(x_i)\eta) \otimes_A b\big) \\
&\quad = \sum_i^n \psi(a_i c)b \otimes L(x_i)\eta \\
&\quad = \sum_i^n (\psi(a_i) \otimes L(x_i))(\psi(c)b \otimes \eta) \\
&\quad = (\psi \otimes L)\Big(\sum_i^n a_i \otimes x_i\Big)(\psi(c)b \otimes \eta) \\
&\quad \approx (\psi \otimes L)\big(\delta_A(a)(1 \otimes x)\big)(\psi(c)b \otimes \eta) \\
&\quad = (\psi \otimes L)(\delta_A(a))(1 \otimes L(x))(\psi(c)b \otimes \eta) \\
&\quad = (\psi \otimes L) \circ \delta_A(a)(\psi(c)b \otimes \xi) \\
&\quad = (\mathrm{id} \otimes L) \circ \delta_B(\psi(a))\Phi((c \otimes \xi) \otimes_B b) \\
&\quad = \pi_L^B(\psi(a))\big(\Phi((c \otimes \xi) \otimes_B b)\big),
\end{aligned}
$$

so that $(\psi \times_r \hat{S}_V)(\pi_L^A(a)) = \pi_L^B(\psi(a))$. Since it is straightforward to check that $(\psi \times_r \hat{S}_V)(1 \otimes \rho(y)) = 1 \otimes \rho(y)$ for $y \in \hat{S}_V$, this shows that $\psi \times_r \hat{S}_V$ maps $A \times_r \hat{S}_V$ into $M(B \times_r \hat{S}_V) \subseteq \mathcal{L}_B(B \otimes H)$ and also establishes Equation (1.3), which in turn makes it evident that $\psi \times_r \hat{S}_V$ is nondegenerate. $\square$

2. Coactions on right-Hilbert bimodules

Let V be a regular multiplicative unitary. For simplicity, we'll just write S for S_V and $\hat{S}$ for $\hat{S}_V$. We define a *coaction* of the Hopf C^*-algebra S on a right-Hilbert $A - B$ bimodule X to be an imprimitivity bimodule coaction $(\delta_K, \delta_X, \delta_B)$ of S on ${}_K X_B$ ([16, Definition 3.3(a)]) together with a C^*-coaction δ_A of S on A such that the associated homomorphism $\psi\colon A \to M(K)$ is $\delta_A - \delta_K$ equivariant. We say that a right-Hilbert bimodule coaction

$(\delta_A, \delta_X, \delta_B)$ is *injective* if δ_A and δ_B are (in which case δ_X will be also), and we say it is *nondegenerate* if δ_A and δ_B are nondegenerate C^*-coactions.

Given an imprimitivity bimodule coaction $(\delta_K, \delta_X, \delta_B)$ of S on ${}_KX_B$, the rule

$$\delta_L \begin{pmatrix} k & x \\ \tilde{y} & b \end{pmatrix} = \begin{pmatrix} \delta_K(k) & \delta_X(x) \\ \delta_X(y)\tilde{} & \delta_B(b) \end{pmatrix}$$

defines a coaction δ_L of S on the linking algebra $L(X) = \left(\begin{smallmatrix} K & X \\ \tilde{X} & B \end{smallmatrix}\right)$ ([16, Lemma 3.7]). Departing slightly from Ng, we will define the *imprimitivity bimodule crossed product* ${}_KX_B \times_{\delta_X} \hat{S}$ to be the corner $j_L(p)(L(X) \times_{\delta_L} \hat{S})j_L(q)$, where $p = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and $q = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ are the canonical projections in $M(L(X))$. By [16, Theorem 3.11], ${}_KX_B \times_{\delta_X} \hat{S}$ is then a $K \times_{\delta_K} \hat{S}$ – $B \times_{\delta_B} \hat{S}$ imprimitivity bimodule which is an imprimitivity bimodule crossed product in Ng's sense ([16, Definition 3.5(b)]), and we have

$$L(X) \times_{\delta_L} \hat{S} \cong \begin{pmatrix} K \times_{\delta_K} \hat{S} & X \times_{\delta_X} \hat{S} \\ (X \times_{\delta_X} \hat{S})\tilde{} & B \times_{\delta_B} \hat{S} \end{pmatrix} = L(X \times_{\delta_X} \hat{S}).$$

Similarly (this time in keeping with Ng), we define the *reduced crossed product* ${}_KX_B \times_{\delta_X,r} \hat{S}$ to be $(\mathrm{id} \otimes L) \circ \delta_L(p)(L(X) \times_{\delta_L,r} \hat{S})(\mathrm{id} \otimes L) \circ \delta_L(q)$ ([16, Remark 3.20(a)]). By the proof of [16, Proposition 3.19], it is a $K \times_{\delta_K,r} \hat{S}$ – $B \times_{\delta_B,r} \hat{S}$ imprimitivity bimodule, and

$$L(X) \times_{\delta_L,r} \hat{S} \cong \begin{pmatrix} K \times_{\delta_K,r} \hat{S} & X \times_{\delta_X,r} \hat{S} \\ (X \times_{\delta_X,r} \hat{S})\tilde{} & B \times_{\delta_B,r} \hat{S} \end{pmatrix} = L(X \times_{\delta_X,r} \hat{S}).$$

Now given a right-Hilbert bimodule coaction $(\delta_A, \delta_X, \delta_B)$ of S on ${}_AX_B$, the nondegenerate homomorphism $\psi \times \hat{S}\colon A \times \hat{S} \to M(K \times \hat{S})$ makes ${}_KX_B \times \hat{S}$ into a right-Hilbert $A \times \hat{S} - B \times \hat{S}$ bimodule, which we denote by ${}_AX_B \times \hat{S}$ and call the *right-Hilbert bimodule crossed product* of ${}_AX_B$ by S. Similarly, the nondegenerate homomorphism $\psi \times_r \hat{S}\colon A \times_r \hat{S} \to M(K \times_r \hat{S})$ of Lemma 1.1 makes ${}_KX_B \times_r \hat{S}$ into a right-Hilbert $A \times_r \hat{S}$ – $B \times_r \hat{S}$ bimodule, which we denote ${}_AX_B \times_r \hat{S}$. If V comes from a Kac system, we define right-Hilbert bimodule coactions of $\hat{S}_V \cong S_{\hat{V}}$ and the right-Hilbert bimodule crossed products ${}_AX_B \times S_V$ and ${}_AX_B \times_r S_V$ by replacing V with $\hat{V}$ in the above definitions.

If $(\delta_K, \delta_X, \delta_B)$ is an imprimitivity bimodule coaction of S on ${}_KX_B$, it is straightforward to check that the dual coaction $\hat{\delta}_L$ of $\hat{S}_p$ on $L(X) \times \hat{S}$ restricts to the dual coactions $\hat{\delta}_K$ and $\hat{\delta}_B$ on the diagonal corners $K \times \hat{S}$ and $B \times \hat{S}$. The restriction of $\hat{\delta}_L$ to the upper right corner ${}_KX_B \times \hat{S}$ gives a map $\hat{\delta}_X$ such that $(\hat{\delta}_K, \hat{\delta}_X, \hat{\delta}_B)$ is an imprimitivity bimodule coaction of $\hat{S}_p$ on ${}_KX_B \times \hat{S}$ which we call the *dual imprimitivity bimodule coaction*. (One can show that this definition agrees with that given for Hilbert modules in [16, Remark 2.18].) If $(\delta_A, \delta_X, \delta_B)$ is a right-Hilbert bimodule coaction of S on ${}_AX_B$, we define the *dual coaction* of $\hat{S}_p$ on ${}_AX_B \times \hat{S}$ to be the dual imprimitivity bimodule coaction $(\hat{\delta}_K, \hat{\delta}_X, \hat{\delta}_B)$, together with the dual

C^*-coaction $\hat{\delta}_A$. Since

$$\begin{aligned}
&((\psi \times \hat{S}) \otimes \mathrm{id}) \circ \hat{\delta}_A(j_A(a)\mu_A(y)) \\
&\quad = ((\psi \times \hat{S}) \otimes \mathrm{id})\big((j_A(a) \otimes 1)(\mu_A \otimes \mathrm{id})(\hat{\delta}_V(y))\big) \\
&\quad = (j_K(\psi(a)) \otimes 1)(\mu_K \otimes \mathrm{id})(\hat{\delta}_V(y)) \\
&\quad = \hat{\delta}_K(j_K(\psi(a))\mu_K(y)) \\
&\quad = \hat{\delta}_K \circ (\psi \times \hat{S})(j_A(a)\mu_A(y))
\end{aligned}$$

for all $a \in A$, $y \in \hat{S}_p$, the nondegenerate homomorphism $\psi \times \hat{S}\colon A \times \hat{S} \to M(K \times \hat{S})$ is $\hat{\delta}_A - \hat{\delta}_K$ equivariant, so this is indeed a right-Hilbert bimodule coaction.

Given a right-Hilbert bimodule coaction $(\delta_A, \delta_K, \delta_K)$ of S on a standard bimodule ${}_AK_K$, we have potentially two different right-Hilbert $A \times \hat{S} - K \times \hat{S}$ bimodules: the bimodule crossed product ${}_AK_K \times \hat{S}$ and the standard bimodule formed from the C^*-algebra crossed product $K \times \hat{S}$ and the nondegenerate homomorphism $\psi \times \hat{S}\colon A \times \hat{S} \to M(K \times \hat{S})$. The following lemma shows that these coincide; in other words, a crossed product of a standard bimodule is a standard bimodule.

Lemma 2.1. *Let V be a regular multiplicative unitary, and let $(\delta_A, \delta_K, \delta_K)$ be a right-Hilbert bimodule coaction of $S = S_V$ on a standard bimodule ${}_AK_K$. Then the right-Hilbert bimodule crossed product ${}_AK_K \times_{\delta_K} \hat{S}$ is isomorphic to the C^*-crossed product $K \times_{\delta_K} \hat{S}$ as a right-Hilbert $A \times_{\delta_A} \hat{S} - K \times_{\delta_K} \hat{S}$ bimodule. An analogous statement also holds for the reduced crossed products.*

Proof. Since ${}_AK_K \times \hat{S}$ is by definition ${}_KK_K \times \hat{S}$ with the same homomorphism $\psi \times \hat{S}$, it suffices to show that ${}_KK_K \times \hat{S}$ is isomorphic to the C^*-crossed product $K \times_{\delta_K} \hat{S}$ as a $K \times_{\delta_K} \hat{S} - K \times_{\delta_K} \hat{S}$ imprimitivity bimodule. Let $L = \left(\begin{smallmatrix} K & K \\ K & K \end{smallmatrix}\right)$ be the linking algebra for ${}_KK_K$, and let $\delta_L = \left(\begin{smallmatrix} \delta_K & \delta_K \\ \delta_K & \delta_K \end{smallmatrix}\right)$ be the associated coaction. Then by [16, Theorem 3.11],

$$L \times_{\delta_L} \hat{S} \cong \begin{pmatrix} K \times \hat{S} & {}_KK_K \times \hat{S} \\ {}_KK_K \times \hat{S} & K \times \hat{S} \end{pmatrix}, \tag{2.1}$$

and the projection $j\left(\begin{smallmatrix} 1_K & 0 \\ 0 & 0 \end{smallmatrix}\right)$ in $M(L \times_{\delta_L} \hat{S})$ corresponds to $\left(\begin{smallmatrix} 1_{K \times \hat{S}} & 0 \\ 0 & 0 \end{smallmatrix}\right)$ under this isomorphism.

Let M_2 denote the C^*-algebra of two-by-two matrices over $\mathbb{C}$. Then the canonical isomorphism $\Phi\colon M_2 \otimes K \to L$ determined by $\Phi\left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \otimes k\right) = \left(\begin{smallmatrix} ak & bk \\ ck & dk \end{smallmatrix}\right)$ is clearly $\mathrm{id} \otimes \delta_K - \delta_K$ equivariant; thus

$$L \times_{\delta_L} \hat{S} \cong (M_2 \otimes K) \times_{\mathrm{id} \otimes \delta_K} \hat{S}. \tag{2.2}$$

Note that this isomorphism takes $j\left(\begin{smallmatrix} 1_K & 0 \\ 0 & 0 \end{smallmatrix}\right) \in M(L \times_{\delta_L} \hat{S})$ to $j\left(\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) \otimes 1_K\right) \in M((M_2 \otimes K) \times_{\mathrm{id} \otimes \delta_K} \hat{S})$.

We next claim that

$$(M_2 \otimes K) \times_{\mathrm{id} \otimes \delta_K} \hat{S} \cong M_2 \otimes (K \times_{\delta_K} \hat{S}). \tag{2.3}$$

For if ι denotes the trivial coaction of $\mathbb{C}$ on M_2, then [17, Proposition 3.2] implies that

$$(M_2 \otimes K) \times_{\iota \otimes \delta_K} \widehat{\mathbb{C} \otimes S} \cong (M_2 \times_\iota \hat{\mathbb{C}}) \otimes (K \times_{\delta_K} \hat{S}),$$

and the right and left sides of this equation are naturally isomorphic to the right and left sides, respectively, of Equation (2.3). Under this isomorphism, the projection $j((\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}) \otimes 1_K) \in M((M_2 \otimes K) \times_{\mathrm{id} \otimes \delta_K} \hat{S})$ is carried to $(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}) \otimes 1_{K \times \hat{S}} \in M(M_2 \otimes (K \times_{\delta_K} \hat{S}))$.

Combining Equations (2.1), (2.2), and (2.3), we have

$$\begin{pmatrix} K \times \hat{S} & {}_K K_K \times \hat{S} \\ {}_K K_K \times \hat{S} & K \times \hat{S} \end{pmatrix} \cong M_2 \otimes (K \times_{\delta_K} \hat{S}),$$

and since $j \left(\begin{smallmatrix} 1_K & 0 \\ 0 & 0 \end{smallmatrix}\right)$ maps to $(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}) \otimes 1$, it follows that the corners ${}_K K_K \times_{\delta_K} \hat{S}$ and $K \times_{\delta_K} \hat{S}$ are isomorphic as $K \times_{\delta_K} \hat{S} - K \times_{\delta_K} \hat{S}$ imprimitivity bimodules.

For the reduced crossed products, it again suffices to show that ${}_K K_K \times_r \hat{S}$ is isomorphic to $K \times_r \hat{S}$ as a $K \times_r \hat{S} - K \times_r \hat{S}$ imprimitivity bimodule. By [16, Remark 3.20(a)] we have

$$L \times_{\delta_L, r} \hat{S} \cong \begin{pmatrix} K \times_r \hat{S} & {}_K K_K \times_r \hat{S} \\ {}_K K_K \times_r \hat{S} & K \times_r \hat{S} \end{pmatrix},$$

and it follows from equivariance of $\Phi\colon M_2 \otimes K \to L$ that

$$L \times_{\delta_L, r} \hat{S} \cong (M_2 \otimes K) \times_{\mathrm{id} \otimes \delta_K, r} \hat{S}.$$

Applying [17, Proposition 3.3], which is the reduced version of [17, Proposition 3.2], we get

$$(M_2 \otimes K) \times_{\iota \otimes \delta_K, r} \widehat{\mathbb{C} \otimes S} \cong (M_2 \times_{\iota, r} \hat{\mathbb{C}}) \otimes (K \times_{\delta_K, r} \hat{S}),$$

and hence

$$(M_2 \otimes K) \times_r \hat{S} \cong M_2 \otimes (K \times_r \hat{S}).$$

Combining these isomorphisms and matching up the projections as above, it follows that ${}_K K_K \times_{\delta_K, r} \hat{S} \cong K \times_{\delta_K, r} \hat{S}$. □

For the proof of our main result (Theorem 4.1) we will need to know that the decomposition ${}_A X_B \cong {}_A K \otimes_K X_B$ is equivariant:

Lemma 2.2. *Let V be a regular multiplicative unitary, let $(\delta_A, \delta_X, \delta_B)$ be a right-Hilbert bimodule coaction of $S = S_V$ on ${}_A X_B$, and let δ_K be the associated coaction on $K = \mathcal{K}_B(X)$. Then there exist right-Hilbert bimodule isomorphisms*

$${}_A X_B \times_{\delta_X} \hat{S} \cong ({}_A K_K \times_{\delta_K} \hat{S}) \otimes_{K \times \hat{S}} ({}_K X_B \times_{\delta_X} \hat{S})$$

and

$${}_A X_B \times_{\delta_X, r} \hat{S} \cong ({}_A K_K \times_{\delta_K, r} \hat{S}) \otimes_{K \times_r \hat{S}} ({}_K X_B \times_{\delta_X, r} \hat{S}).$$

Proof. By definition, ${}_AX_B \times_{\delta_X} \hat{S}$ is the imprimitivity bimodule ${}_KX_B \times_{\delta_X} \hat{S}$ with the nondegenerate homomorphism $\psi \times \hat{S}\colon A \times \hat{S} \to M(K \times \hat{S})$ arising from $\psi\colon A \to M(K)$. Since ${}_AK_K \times \hat{S}$ is ${}_KK_K \times \hat{S}$ with the same map, for the first isomorphism it suffices to show that ${}_KX_B \times_{\delta_X} \hat{S} \cong ({}_KK_K \times_{\delta_K} \hat{S}) \otimes_{K\times\hat{S}} ({}_KX_B \times_{\delta_X} \hat{S})$ as imprimitivity bimodules. But by Lemma 2.1, ${}_KK_K \times \hat{S} \cong K \times \hat{S}$, so the result follows from the usual cancelation $C \otimes_C Y \cong Y$.

The assertion about the reduced crossed products follows similarly from Lemma 2.1. □

3. The fixed-point theorem

Based upon the familiar results for actions of compact groups and coactions of discrete groups, one would guess that the crossed product by a coaction of a Hopf C^*-algebra of compact type is Morita equivalent to the fixed-point algebra. In [18] Ng proves a version of this fixed-point theorem, and this is crucial for his imprimitivity theorem, which we study in the next section. (We should point out that the imprimitivity theorem naturally involves a coaction of $\hat{S}_U$ for a multiplicative unitary U coming from a *discrete* Kac system, but is proved by applying the fixed-point theorem to the corresponding coaction of $S_{\hat{U}}$, where $\hat{U}$ is compact.) Here we recall Ng's fixed-point result and establish some relations to multipliers and bimodules, in preparation for our work with Ng's imprimitivity theorem in Section 4.

Let V be a regular multiplicative unitary of compact type such that S_V has a faithful Haar state φ. Again we'll just write S for S_V and $\hat{S}$ for $\hat{S}_V$. Let δ be a coaction of S on a C^*-algebra A which is *effective* in the sense that

$$\overline{\operatorname{span}}\{\delta(A)(A \otimes 1)\} = A \otimes S.$$

Ng shows in two steps ([18, Theorem 2.7 and Proposition 2.9]) that the reduced crossed product $A \times_{\delta,r} \hat{S}$ is Morita equivalent to the fixed-point algebra A^δ. Since it will simplify our computations with the imprimitivity bimodule, we will combine Ng's two steps into one.

Ng's strategy is to use a nonunital version of Watatani's C^*-basic construction ([22]). The map $E = E_A = (\mathrm{id} \otimes \varphi) \circ \delta$ is a conditional expectation of A onto A^δ, and so A becomes a full pre-Hilbert A^δ-module under right multiplication and the pre-inner product

$$\langle a, b\rangle_{A^\delta} = E(a^*b).$$

The Hausdorff completion of the pre-Hilbert A^δ-module A is a full Hilbert A^δ-module, denoted $\mathcal{F} = \mathcal{F}(A)$. Let $\eta = \eta_A$ be the canonical map of A into $\mathcal{F}$, and define $e_A \in \mathcal{L}_{A^\delta}(\mathcal{F})$ and $\lambda = \lambda_A\colon A \to \mathcal{L}_{A^\delta}(\mathcal{F})$ by

$$e_A\eta(a) = \eta(E(a)) \quad \text{and} \quad \lambda(a)\eta(b) = \eta(ab).$$

Then the C^**-basic construction* is defined to be the closed span in $\mathcal{L}_{A^\delta}(\mathcal{F})$ of $\lambda(A)e_A\lambda(A)$, and is denoted $C^*\langle A, e_A\rangle$. Since

$$e_A\lambda(a)e_A = \lambda(E(a))e_A \quad \text{and} \quad E(a^*) = E(a)^*,$$

$C^*\langle A, e_A\rangle$ is a C^*-algebra; in fact, a routine computation shows $C^*\langle A, e_A\rangle$ coincides with the imprimitivity algebra $\mathcal{K}_{A^\delta}(\mathcal{F})$. Moreover, a short computation shows that the left inner product is given on the generators by

$${}_{C^*\langle A,e_A\rangle}\langle \eta(a), \eta(b)\rangle = \lambda(a)e_A\lambda(b^*).$$

Therefore, the Hausdorff completion $\mathcal{F}$ of the $\operatorname{span}\{\lambda(A)e_A\lambda(A)\} - A^\delta$ pre-imprimitivity bimodule A is a $C^*\langle A, e_A\rangle - A^\delta$ imprimitivity bimodule.

Ng's first step is to temporarily assume the coaction δ is injective. Then the conditional expectation E is faithful, and Ng proves ([18, Theorem 2.7]) that in this case the map

$$\lambda(a)e_A\lambda(b) \mapsto \delta(a)(1 \otimes \rho(\varphi))\delta(b)$$

extends to an isomorphism of the C^*-basic construction $C^*\langle A, e_A\rangle$ onto the reduced crossed product $A \times_{\delta,r} \hat{S}$, where

$$\rho(\varphi) = (\mathrm{id} \otimes \varphi)(V),$$

which, as Ng observes in [18, proof of Lemma 2.5], is a member of $\hat{S}$.

Ng's second step is to remove the injectivity condition on δ and note that, if we put $I = \ker\delta$, there is an injective coaction δ' on A/I given by $\delta'(q(a)) = (q \otimes \mathrm{id}) \circ \delta(a)$, where $q\colon A \to A/I$ is the quotient map. Then δ' is also effective, q maps A^δ isomorphically onto $(A/I)^{\delta'}$, and the reduced crossed products $A \times_{\delta,r} \hat{S}$ and $(A/I) \times_{\delta',r} \hat{S}$ coincide. Ng deduces as a corollary ([18, Proposition 2.9]) that $A \times_{\delta,r} \hat{S}$ is still Morita equivalent to A^δ.

To combine Ng's two steps, note that in the second step the imprimitivity bimodule $\mathcal{F}(A/I)$ is the completion of A/I with inner product

$$\langle q(a), q(b)\rangle_{(A/I)^{\delta'}} = E_{A/I}(q(a)^*q(b)) = q \circ E_A(a^*b) = q\big(\langle a, b\rangle_{A^\delta}\big).$$

Since q is faithful on the image of $\langle\cdot,\cdot\rangle_{A^\delta}$, $\mathcal{F}(A/I)$ can be identified with $\mathcal{F}(A)$. More precisely, the map $\eta_A(a) \mapsto \eta_{A/I}(q(a))$ is well-defined and extends to an isomorphism Φ of the Hilbert A^δ-module $\mathcal{F}(A)$ onto the Hilbert $(A/I)^{\delta'}$-module $\mathcal{F}(A/I)$, with right coefficient map $q|_{A^\delta}$. Moreover, a short computation shows

$$\Phi\big(\lambda_A(a)e_A\lambda_A(b)\eta_A(c)\big) = \lambda_{A/I}(q(a))e_{A/I}\lambda_{A/I}(q(b))\Phi(\eta_A(c)),$$

so Φ is in fact an isomorphism of the $C^*\langle A, e_A\rangle - A^\delta$ imprimitivity bimodule $\mathcal{F}(A)$ onto the $C^*\langle A/I, e_{A/I}\rangle - (A/I)^{\delta'}$ imprimitivity bimodule $\mathcal{F}(A/I)$, with left coefficient map determined by

$$\lambda_A(a)e_A\lambda_A(b) \mapsto \lambda_{A/I}(q(a))e_{A/I}\lambda_{A/I}(q(b)).$$

Combining with the isomorphism

$$\lambda_{A/I}(q(a))e_{A/I}\lambda_{A/I}(q(b)) \mapsto \delta'(q(a))(1 \otimes \rho(\varphi))\delta'(q(b))$$

of $C^*\left\langle A/I, e_{A/I}\right\rangle$ onto $(A/I)\times_{\delta',r}\hat{S}$, and with the identification of $A\times_{\delta,r}\hat{S}$ and $(A/I)\times_{\delta',r}\hat{S}$, we get an isomorphism

$$\lambda_A(a)e_A\lambda(b)\mapsto\delta(a)(1\otimes\rho(\varphi))\delta(b)$$

of $C^*\langle A, e_A\rangle$ onto $A\times_{\delta,r}\hat{S}$. Putting all this together, we have a one-step version of Ng's fixed-point theorem ([18, Proposition 2.11]) — although we haven't addressed the case of coactions by S_p:

Proposition 3.1 ([18])**.** *If V is a regular multiplicative unitary of compact type such that $S=S_V$ has a faithful Haar state φ, and if δ is an effective coaction of S on A, then A is a pre-imprimitivity bimodule between the pre-C^*-algebra $B=\operatorname{span}\{\delta(A)(1\otimes\rho(\varphi))\delta(A)\}$ and the fixed-point algebra A^δ, with operations given for $a,b,c\in A$ and $d\in A^\delta$ by*

$$\begin{aligned}\big(\delta(a)(1\otimes\rho(\varphi))\delta(b)\big)\cdot c&=aE(bc)\\ a\cdot d&=ad\\ {}_B\langle a,b\rangle&=\delta(a)(1\otimes\rho(\varphi))\delta(b^*)\\ \langle a,b\rangle_{A^\delta}&=E(a^*b).\end{aligned}$$

Consequently, the Hausdorff completion $\mathcal{F}(A)$ of A is an $A\times_{\delta,r}\hat{S}$ – A^δ imprimitivity bimodule.

Now let $(\delta_A,\delta_X,\delta_B)$ be a coaction of S on an $A-B$ imprimitivity bimodule X, let $L=L(X)$ be the linking algebra, and let δ_L be the associated coaction of S on L. Then we have

$$\begin{aligned}\delta_L(L)(L\otimes 1)&\supseteq\begin{pmatrix}\delta_A(A)&\delta_X(X)\\ \delta_X(X)\tilde{}&\delta_B(B)\end{pmatrix}\begin{pmatrix}A\otimes 1&0\\ 0&B\otimes 1\end{pmatrix}\\ &=\begin{pmatrix}\delta_A(A)(A\otimes 1)&\delta_X(X)\cdot(B\otimes 1)\\ \delta_X(X)\tilde{}\cdot(A\otimes 1)&\delta_B(B)(B\otimes 1)\end{pmatrix}\end{aligned}$$

and

$$\begin{aligned}\delta_X(X)\cdot(B\otimes 1)&=\delta_X(X\cdot B)\cdot(B\otimes 1)\\ &=\delta_X(X)\cdot\delta_B(B)(B\otimes 1);\end{aligned}$$

it follows from this (and by symmetry) that δ_L is effective whenever δ_A and δ_B are.

Let $p=\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)\in M(L)$. We will need the following result in the next section.

Lemma 3.2. *Let V be a regular multiplicative unitary of compact type such that $S=S_V$ has a faithful Haar state φ, and with notation as above, suppose that δ_A and δ_B are effective. Then the inclusion $A\hookrightarrow L$ extends to an isomorphism Φ of the $A\times_{\delta_A,r}\hat{S}$ – A^{δ_A} imprimitivity bimodule $\mathcal{F}(A)$ onto the $\delta_L(p)(L\times_{\delta_L,r}\hat{S})\delta_L(p)$ – $pL^{\delta_L}p$ imprimitivity bimodule $\delta_L(p)\cdot\mathcal{F}(L)\cdot p$.*

Proof. Let's first make sure we understand all the components of the statement of the lemma. On the right side of $\mathcal{F}(L)$ we regard p as an element

of $M(L^{\delta_L})$, which naturally embeds in $M(L)$. Since the projections p in $M(L^{\delta_L})$ and $\delta_L(p) = p \otimes 1$ in $M(L \times_r \hat{S})$ are full, $\delta_L(p) \cdot \mathcal{F}(L) \cdot p$ is indeed a $\delta_L(p)(L \times_r \hat{S})\delta_L(p) - pL^{\delta_L}p$ imprimitivity bimodule. We have

$$\begin{aligned}
\delta_L(p)(L \times_r \hat{S})\delta_L(p) &= (p \otimes 1)\overline{\operatorname{span}}\{\delta_L(L)(1_{M(L)} \otimes \hat{S})\}(p \otimes 1)\\
&= \overline{\operatorname{span}}\{(p \otimes 1)\delta_L(L)(p \otimes 1)(1_{M(A)} \otimes \hat{S})\}\\
&= \overline{\operatorname{span}}\{\delta_L(p)\delta_L(L)\delta_L(p)(1_{M(A)} \otimes \hat{S})\}\\
&= \overline{\operatorname{span}}\{\delta_L(pLp)(1_{M(A)} \otimes \hat{S})\}\\
&= \overline{\operatorname{span}}\{\delta_A(A)(1_{M(A)} \otimes \hat{S})\}\\
&= A \times_r \hat{S},
\end{aligned}$$

where we have used $p = 1_{M(A)}$. On the other side, since $E_L|_A = E_A$ and the natural extension of E_L to $M(L)$ is a conditional expectation onto $M(L^{\delta_L})$, we have

$$pL^{\delta_L}p = pE_L(L)p = E_L(pLp) = E_A(A) = A^{\delta_A}.$$

Thus, it suffices to show the inclusion $A \hookrightarrow L$ respects the right inner products and the left module multiplications. For the inner products, if $a, b \in A$ then

$$\langle a, b\rangle_{L^{\delta_L}} = E_L(a^*b) = E_A(a^*b) = \langle a, b\rangle_{A^{\delta_A}}.$$

Turning to the left module multiplications, first note that

$$\delta_L(p)(1_{M(L)} \otimes \rho(\varphi))\delta_L(p) = p \otimes \rho(\varphi) = 1_{M(A)} \otimes \rho(\varphi),$$

so for $a, b \in A$ we have

$$\delta_L(a)(1_{M(L)} \otimes \rho(\varphi))\delta_L(b) = \delta_A(a)(1_{M(A)} \otimes \rho(\varphi))\delta_A(b).$$

Hence, for $a, b, c \in A$ we have

$$\begin{aligned}
\big(\delta_L(a)(1_{M(L)} \otimes \rho(\varphi))\delta_L(b)\big) \cdot c &= aE_L(bc) = aE_A(bc)\\
&= \big(\delta_A(a)(1_{M(A)} \otimes \rho(\varphi))\delta_A(b)\big) \cdot c,
\end{aligned}$$

and we're done. □

We'll also need the following lemma concerning standard bimodules.

Lemma 3.3. *Let V be a regular multiplicative unitary of compact type such that $S = S_V$ has a faithful Haar state φ. If $\psi\colon A \to M(B)$ is a nondegenerate homomorphism which is equivariant for effective coactions δ_A and δ_B of S, then ψ extends to a nondegenerate imprimitivity bimodule homomorphism $\Psi\colon \mathcal{F}(A) \to M(\mathcal{F}(B))$ with coefficient maps $\psi \times_r \hat{S}$ and $\psi|_{A^{\delta_A}}$.*

Proof. By [12, Lemma 5.1], it's enough to show ψ preserves both module multiplications and inner products. For $a, b, c \in A$ and $d \in A^{\delta_A}$ we have

$$\begin{aligned}\psi\Big(\big(\delta_A(a)(1 \otimes \rho(\varphi))\delta_A(b)\big) \cdot c\Big) &= \psi(aE_A(bc)) = \psi(a)\psi \circ E_A(bc)\\ &= \psi(a)E_B \circ \psi(bc) = \psi(a)E_B\big(\psi(b)\psi(c)\big)\\ &= \big(\delta_B(\psi(a))(1 \otimes \rho(\varphi))\delta_B(\psi(b))\big) \cdot \psi(c)\\ &= (\psi \times_r \hat{S})\big(\delta_A(a)(1 \otimes \rho(\varphi))\delta_A(b)\big) \cdot \psi(c),\end{aligned}$$

$$\psi(a \cdot d) = \psi(ad) = \psi(a)\psi(d) = \psi(a) \cdot \psi(d),$$

$$\begin{aligned}{}_{M(B \times_r \hat{S})}\langle \psi(a), \psi(b)\rangle &= \delta_B(\psi(a))(1 \otimes \rho(\varphi))\delta_B(\psi(b))\\ &= (\psi \times_r \hat{S})\big(\delta_A(a)(1 \otimes \rho(\varphi))\delta_A(b)\big)\\ &= (\psi \times_r \hat{S})\big({}_{A \times_r \hat{S}}\langle a, b\rangle\big),\end{aligned}$$

and

$$\begin{aligned}\langle \psi(a), \psi(b)\rangle_{M(B^{\delta_B})} &= E_B\big(\psi(a)^*\psi(b)\big) = E_B \circ \psi(a^*b)\\ &= \psi \circ E_A(a^*b) = \psi|_{A^{\delta_A}}\big(\langle a, b\rangle_{A^{\delta_A}}\big).\end{aligned}$$

□

4. Equivariance and imprimitivity

We now turn to the result we call Ng's imprimitivity theorem ([18, Theorem 3.4]). This is an analogue, for multiplicative unitaries of *discrete* type (in fact, coming from discrete Kac systems), of Green's and Mansfield's imprimitivity theorems for group actions and coactions, respectively. Our main theorem (Theorem 4.1) says that Ng's theorem is compatible with equivariant right-Hilbert bimodules; we begin by introducing the notation and construction of Ng's imprimitivity bimodule.

Let U, V, and W be multiplicative unitaries coming from discrete Kac systems, and assume W is a normal submultiplicative unitary of V and U is the corresponding quotient (see [18, Definition 3.2]); this implies that there exist surjective Hopf $*$-homomorphisms $L_{V,W} \colon S_V \to S_W$ and $\rho_{V,U} \colon (\hat{S}_V)_p \to (\hat{S}_U)_p$. Thus any coaction δ of S_V on A can be restricted to a coaction $\delta| = (\mathrm{id} \otimes L_{V,W}) \circ \delta$ of S_W on A, and any dual coaction $\hat{\delta}$ of $(\hat{S}_V)_p$ on $A \times_\delta \hat{S}_V$ can be restricted to a coaction $\hat{\delta}| = (\mathrm{id} \otimes \rho_{V,U}) \circ \hat{\delta}$ of $(\hat{S}_U)_p$ on $A \times_\delta \hat{S}_V$. We can pass to the corresponding coaction of the reduced C^*-algebra $\hat{S}_U$ without changing either the crossed product or the fixed-point algebra, and we continue to denote this coaction by $\hat{\delta}|$.

Now assume further that W is amenable, and let $\rho_{W,V} \colon (\hat{S}_W)_p \to (\hat{S}_V)_p$ be the Hopf $*$-homomorphism vouchsafed by the normality of W in V. Ng shows that if δ is nondegenerate, the nondegenerate homomorphism

$$\phi_A = j_A^V \times (\mu_A^V \circ \rho_{W,V}) \colon A \times_{\delta|} \hat{S}_W \to M(A \times_\delta \hat{S}_V)$$

is actually an isomorphism of $A \times_{\delta|} \hat{S}_W$ onto the fixed-point algebra $(A \times_\delta \hat{S}_V)^{\hat{\delta}|}$, and that the restricted dual coaction $\hat{\delta}|$ of $\hat{S}_U$ is effective. Viewing this as an effective coaction of $S_{\hat{U}}$ with $\hat{U}$ compact, Proposition 3.1 provides an $A \times_\delta \hat{S}_V \times_{\hat{\delta}|,r} S_U$ – $(A \times_\delta \hat{S}_V)^{\hat{\delta}|}$ imprimitivity bimodule $\mathcal{F}(A \times_\delta \hat{S}_V)$; using the isomorphism ϕ_A, this becomes an $A \times_\delta \hat{S}_V \times_{\hat{\delta}|,r} S_U$ – $A \times_{\delta|} \hat{S}_W$ imprimitivity bimodule which we denote by $N(A)$.

With notation as below, we define the map $\delta_X|\colon X \to M(X \otimes S_W)$ to be $(\mathrm{id} \otimes L_{V,W}) \circ \delta_X$; it is straightforward to check that then $(\delta_K|, \delta_X|, \delta_B|)$ is an imprimitivity bimodule coaction of S_W on ${}_K X_B$ and that $\psi\colon A \to M(K)$ is $\delta_A|$ – $\delta_K|$ equivariant. We call the resulting right-Hilbert bimodule coaction $(\delta_A|, \delta_X|, \delta_B|)$ of S_W the *restricted coaction* from S_V. The restricted dual right-Hilbert bimodule coaction $(\hat{\delta}_A|, \hat{\delta}_X|, \hat{\delta}_B|)$ of $(\hat{S}_U)_p$ is defined similarly; its reduction to $\hat{S}_U$ is also denoted $(\hat{\delta}_A|, \hat{\delta}_X|, \hat{\delta}_B|)$.

Theorem 4.1. *Let U, V, and W be multiplicative unitaries coming from discrete Kac systems, with W an amenable normal submultiplicative unitary of V and U the corresponding quotient. Let $(\delta_A, \delta_X, \delta_B)$ be an injective, nondegenerate coaction of S_V on a right-Hilbert A – B bimodule X, and suppose that the associated coaction δ_K on the imprimitivity algebra of X is also nondegenerate. Then the diagram*

$$\begin{tikzcd} A \times_{\delta_A} \hat{S}_V \times_{\hat{\delta}_A|,r} S_U \arrow[r, "N(A)"] \arrow[d, "{}_A X_B \times_{\delta_X} \hat{S}_V \times_{\hat{\delta}_X|,r} S_U"'] & A \times_{\delta_A|} \hat{S}_W \arrow[d, "{}_A X_B \times_{\delta_X|} \hat{S}_W"] \\ B \times_{\delta_B} \hat{S}_V \times_{\hat{\delta}_B|,r} S_U \arrow[r, "N(B)"'] & B \times_{\delta_B|} \hat{S}_W \end{tikzcd} \tag{4.1}$$

commutes in the sense that

$$N(A) \otimes_{A \times \hat{S}_W} ({}_A X_B \times \hat{S}_W) \cong ({}_A X_B \times \hat{S}_V \times_r S_U) \otimes_{B \times \hat{S}_V \times_r S_U} N(B)$$

as right-Hilbert $A \times \hat{S}_V \times_r S_U$ – $B \times \hat{S}_W$ bimodules.

Proof. By definition we have an imprimitivity bimodule ${}_K X_B$, a nondegenerate homomorphism $\psi\colon A \to M(K)$, and a coaction δ_K of S_V on $K = \mathcal{K}_B(X)$ such that ψ is δ_A – δ_K equivariant and $(\delta_K, \delta_X, \delta_B)$ is an imprimitivity bimodule coaction of S_V on ${}_K X_B$ which is nondegenerate by assumption. Our strategy will be to prove a version of Diagram (4.1) for the imprimitivity bimodule ${}_K X_B$, a version for the standard bimodule ${}_A K_K$, and then to combine them using the decomposition Lemma 2.2.

First consider the imprimitivity bimodule ${}_K X_B$: let $L = L(X)$ be the linking algebra of X, let $p = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and $q = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ be the canonical projections in $M(L)$, and let δ_L be the associated coaction of S_V on L. Then δ_L is injective since δ_K and δ_B (hence also δ_X) are. Since δ_B is nondegenerate,

we have

$$\begin{aligned}\overline{\operatorname{span}}\{\delta_X(X)\cdot(1\otimes S)\} &= \overline{\operatorname{span}}\{\delta_X(X\cdot B)\cdot(1\otimes S)\}\\ &= \overline{\operatorname{span}}\{\delta_X(X)\cdot\delta_B(B)(1\otimes S)\}\\ &= \overline{\operatorname{span}}\{\delta_X(X)\cdot(B\otimes S)\}\\ &= X\otimes S;\end{aligned}$$

similarly, the nondegeneracy of δ_K implies that $\overline{\operatorname{span}}\{\widetilde{\delta_X(X)}\cdot(1\otimes S)\} = \widetilde{X\otimes S}$. It follows easily that δ_L is nondegenerate as well. Thus, by [18, Theorem 3.4], we have a $K\times_{\delta_K}\hat{S}_V\times_{\hat{\delta}_K|,r}S_U - K\times_{\delta_K|}\hat{S}_W$ imprimitivity bimodule $N(K)$, and an $L\times_{\delta_L}\hat{S}_V\times_{\hat{\delta}_L|,r}S_U - L\times_{\delta_L|}\hat{S}_W$ imprimitivity bimodule $N(L)$. We claim that

(4.2)
$$N(K)\otimes_{K\times\hat{S}_W}({}_KX_B\times\hat{S}_W)\cong({}_KX_B\times\hat{S}_V\times_r S_U)\otimes_{B\times\hat{S}_V\times_r S_U}N(B)$$

as $K\times\hat{S}_V\times_r S_U - B\times\hat{S}_W$ imprimitivity bimodules.

Now $\delta_L| = \begin{pmatrix}\delta_K| & \delta_X|\\ \delta_{\tilde{X}}| & \delta_B|\end{pmatrix}$, so

$$L(X)\times_{\delta_L|}\hat{S}_W\cong L(X\times_{\delta_X|}\hat{S}_W). \tag{4.3}$$

Also, $(L(X)\times\hat{S}_V,(\hat{S}_V)_p,\hat{\delta}_L)\cong(L(X\times\hat{S}_V),(\hat{S}_V)_p,\epsilon_L)$, where $\epsilon_L = \begin{pmatrix}\hat{\delta}_K & \hat{\delta}_X\\ \hat{\delta}_{\tilde{X}} & \hat{\delta}_B\end{pmatrix}$. It follows that the coactions $\hat{\delta}_L|$ and $\epsilon_L| = \begin{pmatrix}\hat{\delta}_K| & \hat{\delta}_X|\\ \hat{\delta}_{\tilde{X}|} & \hat{\delta}_B|\end{pmatrix}$ of $(\hat{S}_U)_p$ are isomorphic, and therefore that their reductions are, so that

(4.4)
$$L(X)\times_{\delta_L}\hat{S}_V\times_{\hat{\delta}_L|,r}S_U\cong L(X\times_{\delta_X}\hat{S}_V)\times_{\epsilon_L|,r}S_U\cong L(X\times_{\delta_X}\hat{S}_V\times_{\hat{\delta}_X|,r}S_U).$$

Let $p_W,q_W\in M(L(X\times\hat{S}_W))$ and $p_U,q_U\in M(L(X\times\hat{S}_V\times_r S_U))$ be the canonical projections. Then using Equations (4.3) and (4.4) to view $N(L)$ as an $L(X\times\hat{S}_V\times_r S_U) - L(X\times\hat{S}_W)$ imprimitivity bimodule, [7, Lemma 4.6] gives us a $K\times\hat{S}_V\times_r S_U - B\times\hat{S}_W$ imprimitivity bimodule isomorphism

$$\begin{aligned}&(p_U\cdot N(L)\cdot p_W)\otimes_{K\times\hat{S}_W}({}_KX_B\times\hat{S}_W)\\ &\qquad\cong({}_KX_B\times\hat{S}_V\times_r S_U)\otimes_{B\times\hat{S}_V\times_r S_U}(q_U\cdot N(L)\cdot q_W).\end{aligned}$$

Thus, in order to establish Equation (4.2) we only need imprimitivity bimodule isomorphisms $p_U\cdot N(L)\cdot p_W\cong N(K)$ and $q_U\cdot N(L)\cdot q_W\cong N(B)$, and by symmetry it suffices to prove the first. Now the isomorphism $L(X\times\hat{S}_V\times_r S_U)\cong L(X\times\hat{S}_V)\times_r S_U$ takes p_U to $\epsilon_L(p_V)$, and the isomorphisms $L(X\times\hat{S}_W)\cong L(X)\times\hat{S}_W\cong(L(X)\times\hat{S}_V)^{\hat{\delta}_L|}\cong L(X\times\hat{S}_V)^{\epsilon_L|}$ carry p_W to p_V. Therefore, Lemma 3.2 (applied to the coaction of $S_{\hat{U}}$ on $L\times_r\hat{S}_V$ equivalent to $\hat{\delta}_L|$) tells us that

$$p_U\cdot N(L)\cdot p_W\cong\epsilon_L(p_V)\cdot\mathcal{F}(L\times_r\hat{S}_V)\cdot p_V\cong\mathcal{F}(K\times_r\hat{S}_V)\cong N(K),$$

which gives Equation (4.2).

Next we consider the standard bimodule ${}_AK_K$ with the right-Hilbert bimodule coaction $(\delta_A, \delta_K, \delta_K)$. We claim that

(4.5)
$$N(A) \otimes_{A\times\hat{S}_W} ({}_AK_K \times \hat{S}_W) \cong ({}_AK_K \times \hat{S}_V \times_r S_U) \otimes_{K\times\hat{S}_V\times_r S_U} N(K)$$

as right-Hilbert $A \times \hat{S}_V \times_r S_U - K \times \hat{S}_W$ bimodules; by [12, Lemma 5.3] and Lemma 2.1, it's enough to show that there is a nondegenerate imprimitivity bimodule homomorphism Ψ from $N(A)$ to $M(N(K))$ with coefficient maps $\psi \times \hat{S}_V \times_r S_U$ and $\psi \times \hat{S}_W$. Applying Lemma 3.3 to the nondegenerate homomorphism $\psi \times \hat{S}_V : A \times \hat{S}_V \to M(K \times \hat{S}_V)$, which is equivariant for the coactions (of $S_{\hat{U}}$ equivalent to) $\hat{\delta}_A|$ and $\hat{\delta}_K|$ of $\hat{S}_U$, we obtain a nondegenerate imprimitivity bimodule homomorphism $\Psi : \mathcal{F}(A \times \hat{S}_V) \to M(\mathcal{F}(K \times \hat{S}_V))$ with coefficient maps $\psi\times\hat{S}_V\times_r S_U$ and $(\psi\times\hat{S}_V)|_{(A\times\hat{S}_V)^{\hat{\delta}_A|}}$. Now by definition, $\psi \times \hat{S}_V = (j_K^V \circ \psi) \times \mu_K^V$, and Ng's isomorphism $\phi_A : A \times \hat{S}_W \to (A \times \hat{S}_V)^{\hat{\delta}_A|}$ is $j_A^V \times (\mu_A^V \circ \rho_{W,V})$. Thus,

$$\begin{aligned}
\phi_K \circ (\psi \times \hat{S}_W) &= (j_K^V \times (\mu_K^V \circ \rho_{W,V})) \circ ((j_K^W \circ \psi) \times \mu_K^W) \\
&= (j_K^V \circ \psi) \times (\mu_K^V \circ \rho_{W,V}) \\
&= \big((j_K^V \circ \psi) \times \mu_K^V\big) \times \big(j_A^V \times (\mu_A^V \circ \rho_{W,V})\big) \\
&= (\psi \times \hat{S}_V) \circ \phi_A.
\end{aligned}$$

This shows that the isomorphisms $\phi_A : A \times \hat{S}_W \to (A \times \hat{S}_V)^{\hat{\delta}_A|}$ and $\phi_K : K \times \hat{S}_W \to (K\times\hat{S}_V)^{\hat{\delta}_K|}$ carry the coefficient map $(\psi\times\hat{S}_V)|_{(A\times\hat{S}_V)^{\hat{\delta}_A|}}$ to $\psi\times\hat{S}_W$; in other words, viewed as a map of $N(A)$ into $M(N(K))$, Ψ is a nondegenerate imprimitivity bimodule homomorphism with coefficient maps $\psi \times \hat{S}_V \times_r S_U$ and $\psi \times \hat{S}_W$, which establishes Equation (4.5).

We now have a prism

$$\begin{tikzcd}
A\times\hat{S}_V\times_r S_U \arrow[rr, "N(A)"] \arrow[dd, "{}_AX_B\times\hat{S}_V\times_r S_U"'] \arrow[dr, "{}_AK_K\times\hat{S}_V\times_r S_U"] & & A\times\hat{S}_W \arrow[dd, "{}_AX_B\times\hat{S}_W"' near end] \arrow[dr, "{}_AK_K\times\hat{S}_W"] & \\
& K\times\hat{S}_V\times_r S_U \arrow[rr, "N(K)"] \arrow[dl, "{}_KX_B\times\hat{S}_V\times_r S_U"] & & K\times\hat{S}_W \arrow[dl, "{}_KX_B\times\hat{S}_W"] \\
B\times\hat{S}_V\times_r S_U \arrow[rr, "N(B)"] & & B\times\hat{S}_W &
\end{tikzcd}$$

in which the front two faces commute by the above arguments, and the commutativity of the back face is the desired result; it only remains to show that the two side triangles commute. That is, we need to know that

$${}_AX_B \times \hat{S}_V \times_r S_U \cong ({}_AK_K \times \hat{S}_V \times_r S_U) \otimes_{K\times\hat{S}_V\times_r S_U} ({}_KX_B \times \hat{S}_V \times_r S_U)$$

and

$$ {}_A X_B \times \hat{S}_W \cong ({}_A K_K \times \hat{S}_W) \otimes_{K\times\hat{S}_W} ({}_K X_B \times \hat{S}_W) $$

as right-Hilbert $A \times \hat{S}_V \times_r S_U$ – $B \times \hat{S}_V \times_r S_U$ and $A \times \hat{S}_W$ – $B \times \hat{S}_W$ bimodules, respectively. But this follows from Lemma 2.2, in the first case applied to the coaction $(\hat{\delta}_A|, \hat{\delta}_X|, \hat{\delta}_B|)$ of $\hat{S}_U$ on ${}_A X_B \times \hat{S}_V$ and then using $K \times \hat{S}_V \cong {}_A K_K \times \hat{S}_V$ from Lemma 2.1. □

Remark 4.2. For an imprimitivity bimodule coaction $(\delta_K, \delta_X, \delta_B)$ of a Hopf C^*-algebra, it is probably true that δ_K is nondegenerate whenever δ_B is; this would simplify the hypotheses of Theorem 4.1 somewhat. Unfortunately, we have been unable to find a proof. (For group coactions, it is true — see [11, Proposition 2.3] — and the proof is fairly nontrivial.) It may be possible to finesse the problem, but since our main point here is to illustrate our approach to imprimitivity theorems, we have chosen not to get mired in nondegeneracy issues.

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